Notes on Black Hole Fluctuations and Backreaction *

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Abstract

In these notes we prepare the ground for a systematic investigation into the issues of black hole fluctuations and backreaction by discussing the formulation of the problem, commenting on possible advantages and shortcomings of existing works, and introducing our own approach via a stochastic semiclassical theory of gravity based on the Einstein-Langevin equation and the fluctuation-dissipation relation for a self-consistent description of metric fluctuations and dissipative dynamics of the black hole with backreaction from its Hawking radiance.

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1 Classical and Semiclassical Backreaction of Metric Perturbations

The idea of viewing a black hole (particle detector) interacting with a quantum field as a dissipative system, and the Hawking [1]- Unruh [2] radiation as a manifestation of a fluctuation-dissipation relation was first proposed by Candelas and Sciama [3, 4]. Even though, as we will soon see, the fluctuations in the thoughts of these earlier authors are not the correct ones and the relations proposed not really addressing the backreaction of quantum fields in a classical black hole spacetime, the idea remains attractive. Indeed one of us (BLH) found it so attractive that he launched a systematic investigation into the statistical mechanical properties of particle/spacetime and quantum field interactions. This involved the introduction of statistical mechanical ideas such as quantum open sytems [5] and field-theoretical methods such as the influence functional [6] and Schwinger-Keldysh formalisms [7] for the establishment of a quantum statistical field theory in curved spacetime (for a review, see [8, 9]). It was found that the backreaction of quantum fields (through processes like particle creation) on a classical background spacetime can be described by an Einstein-Langevin equation [10, 11], which is a generalization of the semiclassical Einstein equation to include stochastic sources due to created particles. It was also found from first principles that the backreaction can be encapsulated in the form of a Fluctuation-Dissipation Relation (FDR) [12, 14], which takes into account the mutual influence of the quantum field and the background spacetime (or detector, in the case of Unruh radiation). We expect it to hold also for black hole systems, both in the familiar static condition where black hole thermodynamics based on the Bekenstein-Hawking relation was constructed, and for dynamical collapse problems. This is the major theme in our current program of research.

Note that it is nontrivial that such a relation exists at least on two counts. First, that the dynamics of spacetime interacting with a quantum field can indeed be treated like a classical particle in a trajectory with stochastic components as depicted in quantum Brownian motion [16, 17]; and second, although in statistical thermodynamics the FDR is usually derived for near-equilibrium conditions (via linear response theory) , these semiclassical gravity processes can serve as an illustration that such a relation can indeed exist for nonequilibrium processes. This was conjectured by one of us in 1989 [8] and demonstrated in succeeding work [10, 11, 12, 13]. We will follow this line of thought to pursue black hole backreaction problems. In this section we will describe in general terms the classical and semiclassical backreaction of metric perturbations. In Sec. 2 we describe stochastic semiclassical gravity in a cosmological setting, focusing on the derivation of an Einstein-Langevin equation. In Sec. 3 we describe metric fluctuations and backreaction in semiclassical gravity, and the FDR for a black hole in equilibrium with its thermal radiation. We comment on the inadequacy in Mottola's [40] derivation of a FDR for backreaction of static black holes. In Sec. 4 we discuss dynamical black holes, identifying the Bardeen metric as a fruitful avenue for semiclassical backreaction analysis. We also comment on the nature of Candelas and Sciama's [3] FDR and its shortcoming for the description of backreaction. We then summarize Bekenstein's theory of black hole fluctuations and note the difference from our approach. In these notes

we are merely preparing the ground for our investigation by sorting out the issues and identifying the inadequacies of previous approaches. Details of our findings will appear in later publications.

For brevity we shall use schematic expressions to discuss the ideas here while relegating the details to research papers in progress. Let us start with the classical Einstein equation for a spacetime with metric $g_{\alpha\beta}$

$$G_{\mu\nu}(g_{\alpha\beta}) = \kappa T^{cm}_{\mu\nu} \tag{1.1}$$

where $G_{\mu\nu}$ is the Einstein tensor, $\kappa = 8\pi G_N$, G_N being the Newton constant, and $T^{cm}_{\mu\nu}$ is the energy momentum tensor of some classical matter (cm). Consider perturbations of the metric tensor at the classical level, i.e.,

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + ... \tag{1.2}$$

where the superscript n in parentheses on $h^{(n)}_{\alpha\beta}$ indicates the order of the perturbation. Most studies of gravitational waves and instability are carried out for linear perturbations $n = 1$.

The linear perturbations (in harmonic gauge) satisfy the linearized Einstein equations in the form [21]

$$\Box h_{\alpha\beta} = 2\kappa\delta T^{cm}_{\alpha\beta} \tag{1.3}$$

The problem of gravitational perturbations in a cosmological Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime in relation to galaxy formation was first treated in detail by Lifshitz [20]. Perturbations in a Schwarzschild black hole spacetime was treated by Regge and Wheeler, Vishveshwara, Zerilli et al [18] and in a Kerr black hole by Teukolsky, Chandrasekhar and others [19].

For quantum fields in a classical background spacetime with metric $g^{(0)}_{\mu\nu}$, the wave equation for, say, a massive (m) minimally coupled scalar field Φ is

$$\Box\Phi + m^2\Phi = 0 \tag{1.4}$$

The first order metric perturbations $h^{(1)}_{\alpha\beta}$ in a vacuum ($T_{\alpha\beta} = 0$) obey an equation similar in form to the above with $m = 0$ (the Lifshitz equation [20]) and can thus be identified as two components (because of the 2 polarizations) of a massless minimally coupled scalar field.

Now let us consider the backreaction of gravitational perturbations or quantum fields in classical and semiclassical gravity. At the classical level, assuming a vacuum background, the linear perturbations $h^{(1)}_{\mu\nu}$ would contribute a source to the $O(\epsilon^0)$ equation in the form [21](Eq. 35.70)

$$G_{\mu\nu}(g^{(0)}) = \kappa\langle T^{gw}_{\mu\nu}\rangle_I \tag{1.5}$$

where

$$T^{gw}_{\mu\nu} \equiv \frac{1}{32\pi}[\bar{h}^{(1)}_{\alpha\beta|\mu}\bar{h}^{(1)}_{\alpha\beta|\nu}], \tag{1.6}$$

is the (inhomogeneous) energy momentum tensor of the gravitational waves (gw) described (in a transverse-traceless gauge) by $\bar{h}^{(1)}_{\alpha\beta} \equiv h^{(1)}_{\alpha\beta} - \frac{1}{2}h^{(1)}g^{(0)}_{\alpha\beta}$. Here the $\langle\,\rangle_I$ around $T^{gw}_{\mu\nu}$ denotes

the Isaacson average over the inhomogeneous sources (taken over some intermediate wavelength range larger than the natural wavelength of the waves but smaller than the curvature radius of the background spacetime). This is an example of backreaction at the classical level. (For related work see [22].)

At the semiclassical level, the backreaction comes from particles created in the quantum field (qf) on the background spacetime [23, 24]. Schematically the semiclassical Einstein equation takes the form

$$G_{\mu\nu}(g^{(0)}) = \kappa \langle T^{qf}_{\mu\nu} \rangle_V \tag{1.7}$$

where

$$T^{qf}_{\mu\nu} \equiv \Phi,_\mu \Phi,_\nu + \frac{1}{2} m^2 \Phi^2 g_{\mu\nu} \tag{1.8}$$

is the energy momentum tensor of, say, a massive minimally coupled scalar field [1]. Here $\langle\,\rangle_V$ denotes expectation value taken with respect to some vacuum state with symmetry commensurate with that of the background spacetime. Studies of semiclassical Einstein equation has been carried out in the last two decades by many authors for cosmological [27] and black hole spacetimes [28].

This is the point where our story begins. In the last decade we have been able to move one step beyond in the semiclassical backreaction problem, extending the above semiclassical framework to a stochastic semiclassical theory, where noise and fluctuations from particle creation are accounted for and incorporated. Spacetime dynamics is now governed by a stochastic generalization of the semiclassical Einstein equation known as the Einstein-Langevin equation — the conventional theory of semiclassical gravity with sources given by the vacuum expectation value of the energy momentum tensor being a mean field approximation of this new theory. We now describe the essence of this new theory, using gravitational perturbations for illustration, again in a schematic form. In this context we can see the distinction between metric perturbations and metric fluctuations on the one hand and the proper meaning of metric fluctuations on the other.

2 Stochastic Semiclassical Gravity: Einstein-Langevin Equation

For concreteness let us consider the background spacetime to be a spatially flat FLRW universe with metric $\tilde{g}^{RW}_{\mu\nu}$ plus small perturbations $\tilde{h}_{\mu\nu}$,

$$\tilde{g}_{\mu\nu}(x) = \tilde{g}^{RW}_{\mu\nu} + \tilde{h}_{\mu\nu} \equiv e^{2\alpha(\eta)} g_{\mu\nu} \tag{2.1}$$

It is conformally related [with conformal factor $\exp(2\alpha(\eta))$] to the Minkowski metric $\eta_{\mu\nu}$ and its perturbations $h_{\mu\nu}(x)$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x) \tag{2.2}$$

[1]By virtue of what we discussed above, the gravitons – quantized linear perturbations of the background metric – obey an equation similar in form to that of a massless minimally coupled quantum scalar field. For graviton production in cosmological spacetimes see [25, 26].

Here η is the conformal time related to the cosmic time t by $dt = \exp[\alpha(\eta)]d\eta$. The perturbations $\tilde{h}_{\mu\nu}$ can be either anisotropic, as in a Bianchi type (Type I case is treated by Hu and Sinha [12]), or inhomogeneous (treated by Campos and Verdaguer [13]). Here we follow these works.

The classical action for a free massless conformally coupled real scalar field $\Phi(x)$ is given by

$$S_f[\tilde{g}_{\mu\nu}, \Phi] = -\frac{1}{2}\int d^n x \sqrt{-\tilde{g}} \left[\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \xi(n)\tilde{R}\Phi^2\right], \tag{2.3}$$

where $\xi(n) = (n-2)/[4(n-1)]$, and $\tilde{R}$ is the Ricci scalar for the metric $\tilde{g}_{\mu\nu}$. Define a conformally related field $\phi(x) \equiv \exp[(n/2-1)\alpha(\eta)]\Phi(x)$, the action S_f (after one integration by parts)

$$S_f[g_{\mu\nu}, \phi] = -\frac{1}{2}\int d^n x \sqrt{-g} \left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \xi(n)R\phi^2\right] \tag{2.4}$$

takes the form of an action for a free massless conformally coupled real scalar field $\phi(x)$ in a spacetime with metric $g_{\mu\nu}$, *i.e.* a nearly flat spacetime. As the physical field $\Phi(x)$ is related to the field $\phi(x)$ by a power of the conformal factor a positive frequency mode of the field $\phi(x)$ in flat spacetime will correspond to a positive frequency mode in the conformally related space. One can thus establish a quantum field theory in the conformally related space by use of the conformal vacuum (see [24]). Quantum effects such as particle creation and trace anomalies arise from the breaking of conformal flatness of the spacetime produced by the perturbations $h_{\mu\nu}(x)$.

The stochastic semiclassical Einstein equation differs from the semiclassical Einstein equation (SCE) by a) the presence of a stochastic term measuring the fluctuations of quantum sources (arising from the difference of particles created in neighboring histories, see, [10]) and b) a dissipation term in the dynamics of spacetime (see the discussion following Eq. (2.7) for earlier treatments of metric dissipation). Thus it endows the form of an Einstein-Langevin equation [11]. Two points are noteworthy: a) The fluctuations and dissipation (kernels) obey a fluctuation -dissipation relation, which embodies the backreaction effects of quantum fields on classical spacetime. b) The stochastic source term engenders metric fluctuations. The semiclassical Einstein equation depicts a mean field theory which one can retrieve from the Einstein-Langevin equation by taking a statistical average with respect to the noise distribution.

The stochastic semiclassical Einstein equation, or Einstein-Langevin equation takes on the form

$$\begin{aligned} \tilde{G}^{\mu\nu}(x) &= \kappa\left(T_c^{\mu\nu} + T_{qs}^{\mu\nu}\right) \\ T_{qs}^{\mu\nu} &\equiv \langle T^{\mu\nu}\rangle_q + T_s^{\mu\nu} \\ T_s^{\mu\nu} &\equiv 2e^{-6\alpha}F^{\mu\nu}[\xi] \end{aligned} \tag{2.5}$$

Here, $T_c^{\mu\nu}$ is due to classical matter or fields, $\langle T^{\mu\nu}\rangle_q$ is the vacuum expectation value of the stress tensor of the quantum field, and $T_{qs}^{\mu\nu}$ is a new stochastic term. Up to first order in $h_{\mu\nu}$

they are given by

$$
\begin{aligned}
\langle T^{\mu\nu}_{(0)}\rangle_q &= \lambda\left[\tilde{H}^{\mu\nu}_{(0)} - \frac{1}{6}\tilde{B}^{\mu\nu}_{(0)}\right] \\
\langle T^{\mu\nu}_{(1)}\rangle_q &= \lambda\Bigg[\ (\tilde{H}^{\mu\nu}_{(1)} - 2\tilde{R}^{(0)}_{\alpha\beta}\tilde{C}^{\mu\alpha\nu\beta}_{(1)}) - \frac{1}{6}\tilde{B}^{\mu\nu}_{(1)} \\
&\qquad +3e^{-6\alpha}\left(-4(C^{\mu\alpha\nu\beta}_{(1)}\alpha)_{,\alpha\beta} + \int d^4y A^{\mu\nu}_{(1)}(y)\mathrm{K}(x-y;\bar{\mu})\right)\ \Bigg].
\end{aligned}
\tag{2.6}
$$

where the constant $\lambda = 1/2880\pi^2$ characterizes one-loop quantum correction terms (which include the trace anomaly and particle creation processes [30]) and $\bar{\mu}$ is a renormalization parameter. Here $C_{\mu\alpha\nu\beta}$ is the Weyl curvature tensor and the tensors $B^{\mu\nu}(x)$, $A^{\mu\nu}(x)$ and $H^{\mu\nu}(x)$ are given by (see, e.g., [13, 29, 30] and earlier references)

$$
\begin{aligned}
B^{\mu\nu}(x) &\equiv \frac{1}{2}g^{\mu\nu}R^2 - 2RR^{\mu\nu} + 2R^{;\mu\nu} - 2g^{\mu\nu}\Box_g R, \\
A^{\mu\nu}(x) &\equiv \frac{1}{2}g^{\mu\nu}C_{\alpha\beta\rho\sigma}C^{\alpha\beta\rho\sigma} - 2R^{\mu\alpha\beta\rho}R^{\nu}{}_{\alpha\beta\rho} + 4R^{\mu\alpha}R_{\alpha}{}^{\nu} \\
&\quad -\frac{2}{3}RR^{\mu\nu} - 2\Box_g R^{\mu\nu} + \frac{2}{3}R^{;\mu\nu} + \frac{1}{3}g^{\mu\nu}\Box_g R, \\
H^{\mu\nu}(x) &\equiv -R^{\mu\alpha}R_{\alpha}{}^{\nu} + \frac{2}{3}RR^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{4}g^{\mu\nu}R^2.
\end{aligned}
\tag{2.7}
$$

All terms mentioned so far in the semiclassical Einstein equation, including the dissipative kernel K, are familiar from backreaction calculations done in the seventies and eighties (see. e.g., [27]). The new result in the nineties is in the appearance of a stochastic source [12, 13], the tensor $F^{\mu\nu}(x)$

$$
F^{\mu\nu}(x) = -2\partial_\alpha\partial_\beta\xi^{\mu\alpha\nu\beta}(x), \tag{2.8}
$$

which is symmetric and traceless, *i.e.* $F^{\mu\nu}(x) = F^{\nu\mu}(x)$ and $F^{\mu}_{\mu}(x) = 0$ (implying that there is no stochastic correction to the trace anomaly). It accounts for the noise associated with fluctuations of the quantum field. For spacetimes with linear metric perturbations as considered here, the stochastic correction to the stress tensor has vanishing divergence (to first order in the metric perturbations).

The stochastic source given by the noise tensor $\xi_{\mu\nu\alpha\beta}(x)$ (which for this problem has the symmetries of the Weyl tensor) is characterized completely by the two point correlation function which is the noise kernel $N(x-y)$ (the probability distribution for the noise is Gaussian) [12, 13]

$$
\begin{aligned}
\langle\xi_{\mu\nu\alpha\beta}(x)\rangle_\xi &= 0, \\
\langle\xi_{\mu\nu\alpha\beta}(x)\xi_{\rho\sigma\lambda\theta}(y)\rangle_\xi &= T_{\mu\nu\alpha\beta\rho\sigma\lambda\theta}\mathrm{N}(x-y),
\end{aligned}
\tag{2.9}
$$

where $T_{\mu\nu\alpha\beta\rho\sigma\lambda\theta}$ is the product of four metric tensors (in such a combination that the right-hand side of the equation satisfies the Weyl symmetries of the two stochastic fields on the left-hand side). Its explicit form is given in [13]

If we now take the mean value of equation (2.5) with respect to the stochastic source ξ we find that, as a consequence of the noise correlation relation,

$$\langle T^{\mu\nu}_{eff} \rangle_\xi = \langle T^{\mu\nu} \rangle_q \tag{2.10}$$

and we recover the semiclassical Einstein equations.

The stochastic term $2F^{\mu\nu}$ will produce a stochastic contribution $h^{st}_{\mu\nu}$ to the spacetime inhomogeneity, *i.e.* $h_{\mu\nu} = h^c_{\mu\nu} + h^{st}_{\mu\nu}$, which we call metric fluctuations. Considering a flat background spacetime for simplicity (setting $\alpha = 0$), one obtains (by adopting the harmonic gauge condition $(h^{st}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{st})^{,\nu} = 0$) a linear equation for the metric fluctuations $h^{st}_{\mu\nu}$

$$\begin{aligned} \Box h^{st}_{\mu\nu} &= 2\kappa S^{st}_{\mu\nu}, \\ S^{\mu\nu}_{st} &= 2F^{\mu\nu} = -4\partial_\alpha\partial_\beta\xi^{\mu\alpha\nu\beta}, \end{aligned} \tag{2.11}$$

The solution of these equations and the computation of the noise correlations have been given by Campos and Verdaguer in their beautiful paper, from where the above schematic description is adapted and further details can be found.

We believe this new framework is fruitful for investigation into metric fluctuations and backreaction effects. We and others [12, 13, 31] have applied it to study quantum effects in cosmological spacetimes. Work on black hole spacetimes is just beginning. Let us first review what has been done before, what we regard as deficient and describe the setup of this problem in our approach.

3 Metric fluctuations and Backreaction in Semiclassical Gravity

It is perhaps useful to begin by emphasizing the difference in meaning of 'metric fluctuations' used in our approach and that used by others. In the glossary of almost all other authors metric fluctuations have been used in a test field context, referring to the two-point function of gravitational perturbations $h_{\mu\nu}$ in the classical sense or the expectation value of graviton two-point function $\langle h_{\mu\nu}(x)h_{\rho\sigma}(y)\rangle$ in a semiclassical sense – semiclassical in that the background remains classical even though the perturbations are quantized. It is in a test field context because one considers gravitational perturbations and their two-point functions from a fixed background geometry . This is a useful concept, but says nothing about backreaction. It is useful as a measure of the fluctuations in the gravitational field at particular regions of spacetime. Ford has explored this aspect in great detail. For example, the recent work of Ford and Svaiter [32] shows that black hole horizon fluctuations are much smaller than Planck dimensions for black holes whose mass exceeds the Planck mass. For these black holes they induced that semiclassical derivation of the Hawking radiance should remain valid, and that contrary to some recent claims [33, 34], there is no drastic effect near the horizon arising from metric fluctuations. However, for backreaction considerations, where the background spactime metric varies in accordance with the behavior of quantum

fields present, the graviton 2-point function calculated with respect to a fixed background is not the relevant quantity to consider.

In contrast, metric fluctuations $h_{\mu\nu}^{st}$ (see Eq.(2.11)) in our work [10, 11, 12] and that of Campos and Verdaguer [13] are defined for semiclassical gravity in a manifestly backreaction context. They are classical stochastic quantities arising from stress tensor fluctuations in the quantum fields present and are perhaps important only at the Planck scale[2]. We see that they are derived from the noise kernel, which involves 4-point functions of the gravitons. It is this quantity which enters into the fluctuation-dissipation relation – not the usual graviton 2 point function – which encapsulates the semiclassical backreaction. This is an important conceptual point which has not been duly recognized.

3.1 Fluctuation-Dissipation Relation Description of Semiclassical Backreaction

With this understanding let us now expound the typical form of fluctuation-dissipation relation (FDR) in quantum field theory, starting with the paradigm of quantum Brownian motion (see, for example, [6]). Consider a quantum mechanical detector or atom coupled linearly to a quantized, otherwise free, field which is initially in some quantum state, pure or mixed (typically taken to be a thermal state). After the coupling is switched on, the atom will "relax" to equilibrium over a time scale which depends on the coupling constant. When the atom reaches equilibrium, the equilibrium fluctuations of its observables depend on the quantum state of the field (for example, if they are thermal fluctuations, the associated temperature will be the temperature of the field). There are therefore two relevant processes: dissipation in the atom as it approaches equilibrium, and fluctuations at equilibrium. These two processes are generally related by a fluctuation- dissipation relation.

For quantum fields, let us consider the model of a simplified atom or detector with internal coordinate Q, coupled to a quantized scalar field ϕ via a bilinear interaction with coupling constant e: $L_I(t) = eQ(t)\phi(x(t))$, t being the atom's proper time and $x(t)$ denoting its parametrized trajectory. It can be shown [11, 15] that the semiclassical dynamics of Q is given by stochastic equations of the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} - \frac{\partial L}{\partial Q} + 2\int^{t} \gamma(t,s)\dot{Q}(s) = \xi(t), \tag{3.1}$$

where $\xi(t)$ is a stochastic force arising out of quantum or thermal fluctuations of the field, and L is the free Lagrangian describing the dynamics of the internal coordinate Q. Its correlator is defined as $\langle \xi(t)\xi(t')\rangle = \hbar\nu(t,t')$.

The functions γ and ν can be written in terms of two-point functions of the field bath

[2] The two point function of gravitons are not stochastic variables and so in a stricter sense they should not be called metric 'fluctuations'. To avoid confusion we may at times called our quantities $h_{\mu\nu}^{st}$, induced metric fluctuations.

surrounding the atom. Thus,

$$\begin{aligned}\mu(t,t') &= \frac{d}{d(t-t')}\gamma(t-t') = \frac{e^2}{2}G(x(t),x(t')) = -i\frac{e^2}{2}\langle[\phi(x(t),\phi(x(t'))]\rangle \\ \nu(t,t') &= \frac{e^2}{2}G^{(1)}(x(t),x(t')) = \frac{e^2}{2}\langle\{\phi(x(t),\phi(x(t'))\}\rangle,\end{aligned} \tag{3.2}$$

where G and $G^{(1)}$ are respectively the Schwinger (commutator) and Hadamard (anticommutator) functions of the free field ϕ evaluated at two points on the atom's trajectory. Both functions are evaluated in whatever quantum state the field is initially in, not necessarily a vacuum state. Because of the way μ and ν enter the equations of motion, they are referred to as *dissipation* and *noise* kernels, respectively. It should be noted that these two kernels, although independent of Q, ultimately determine the rate of energy dissipation of the internal coordinate Q and its quantum/thermal fluctuations.

The statement of the FDR for such cases is that ν and γ are related by a linear non-local relation of the form

$$\nu(t-t') = \int d(s-s')K(t-t',s-s')\gamma(s-s'). \tag{3.3}$$

For thermal states of the field, and in $3+1$ dimensions, it can be shown that

$$K(t,s) = \int_0^\infty \frac{d\omega}{\pi}\omega\coth\left(\frac{1}{2}\beta\hbar\omega\right)\cos(\omega(t-s)), \tag{3.4}$$

where β is the inverse temperature. Equations (3.3) and (3.4) constitute the general form of a FDR. With these ideas in mind, we will now discuss the two forms of FDR which have appeared in the literature in the context of static (this section) and dynamic (next section) black holes.

3.2 Fluctuation and Backreaction in Static Black Holes

Backreaction in this context usually refers to seeking a consistent solution of the semiclassical Einstein equation for the geometry of a black hole in equilibrium with its Hawking radiation (enclosed in a box to ensure relative stability). Much effort in the last 15 years has been devoted to finding a regularized energy-momentum tensor for the backreaction calculation. See [37] for recent status and earlier references. One important early work on backreaction is by York [35], while the most thorough is carried out by Hiscock, Anderson et al [36].

Since the quantum field in such problems is assumed to be in a Hartle-Hawking state, concepts and techniques from thermal field theory are useful. Hartle and Hawking [38], Gibbons and Perry [39] used the periodicity condition of the Green function on the Euclidean section to give a simple derivation of the Hawking temperature for a Schwarzschild black hole. In the same vein, Mottola [40] showed that in some generalized Hartle-Hawking states a FDR exists between the expectation values of the commutator and anti-commutator of the

energy-momentum tensor This FDR is similar to the standard thermal form found in linear response theory:

$$S_{abcd}(x,x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \coth\left(\frac{1}{2}\beta\omega\right) \tilde{D}_{abcd}(\mathbf{x},\mathbf{x}';\omega), \tag{3.5}$$

where S and D are the anticommutator and commutator functions of the energy-momentum tensor, respectively, and $\tilde{D}$ is the temporal Fourier transform of D. That is,

$$\begin{aligned} S_{abcd}(x,x') &= \langle \{\hat{T}_{ab}(x),\, \hat{T}_{cd}(x')\} \rangle_\beta \\ D_{abcd}(x,x') &= \langle [\hat{T}_{ab}(x),\, \hat{T}_{cd}(x')] \rangle_\beta. \end{aligned} \tag{3.6}$$

He also identifies the two-point function D as a dissipation kernel by relating it to the time rate of change of the energy density when the metric is slightly perturbed. Thus, Eq.(3.5) represents a *bona fide* FDR relating the fluctuations of a certain quantity (say, energy density) to the time rate of change of the very same quantity.

However, this type of FDR has rather restricted significance as it is based on the assumption of a fixed background spacetime (static in this case) and state (thermal) of the matter field(s). It is not adequate for the description of backreaction where the spacetime and the state of matter are determined in a self-consistent manner by their dynamics and mutual influence. One should look for a FDR for a parametric family of metrics (belonging to a general class) and a more general state of the quantum matter (in particular, for the Unruh state). We expect the derivation of such an FDR will be far more complicated than the simple case where the Green functions are periodic in imaginary time, and where one can simply take the results of linear response theory almost verbatim.

Even in this simple case, it is noteworthy that there is a small departure from standard linear response theory for quantum systems. This arises from the observation that the dissipation kernel in usual linear response analyses is given by a two-point commutator function of the underlying quantum field, which is independent of the quantum state for free field theory. In this case, we are still restricted to free fields in a curved background. However, since the dissipation now depends on a two-point function of the stress-tensor, it is a four-point function of the field, with appropriate derivatives and coincidence limits. This function is, in general, state-dependent. We have seen examples from related cosmological backreaction problems where it is possible to explicitly relate the dissipation to particle creation in the field, which is definitely a state-dependent process. For the black-hole case, this would imply a quantum-state-dependent damping of semiclassical perturbations. The temperature dependence of the anticommutator function S as displayed in the fluctuation-dissipation relation of Eq. (3.5) is therefore misleading, because the commutator function D can also be temperature-dependent.

To obtain a causal FDR for states more general than the Hartle-Hawking state, one needs to use the in-in (or Schwinger-Keldysh) formalism applied to a class of quasistatic metrics (generalization of York [35]) and proceed in a way leading to the noise kernel similar to that illustrated in Sec. 2. The calculation of 4-point functions of a thermal scalar field in black hole spacetimes [41] and the derivation of such an FDR are in progress [42].

4 Fluctuations and Backreaction in Dynamical Black Holes

4.1 Quasistationary Approximation and Bardeen Metric

Backreaction for dynamical (collapsing) black holes are much more difficult to treat than static ones, and there are fewer viable attempts. For situations with black hole masses much greater than the Planck mass, one important work which captures the overall features of dynamical backreaction is that by Bardeen [43], who, using a generalization of a classical model geometry (Vaidya metric for stars with outgoing perfect fluid), argued that the mass of a radiating black hole decreases at a rate given by its luminosity, as expected from energy considerations. That is,

$$\frac{dM}{dt} = -L. \tag{4.1}$$

In particular, for a black hole emitting Hawking radiation, the luminosity goes as $L = \alpha\hbar M^{-2}$, M being the black hole mass, and α some constant. Far from the horizon ($r > \mathcal{O}(6M)$), Bardeen's geometry takes the form

$$ds^2 = -\left(1 - \frac{2m(u)}{r}\right) du^2 - 2du\, dr + r^2 d\Omega^2, \tag{4.2}$$

with

$$m(u) = \int^u du\, L(u), \tag{4.3}$$

$L(u)$ being the Hawking luminosity, and $m(u)$ the Bondi mass. With this Bardeen argued that the semiclassical picture of black hole evaporation should hold until the black hole reaches Planck size.

More recently Parentani and Piran [44], using a spherically symmetric geometry and a simplified scalar field model which neglects the potential barrier, carried out a numerical integration of the coupled quantized scalar field and semiclassical Einstein equations, and showed that the solution of the semiclassical theory in this model is the geometry described by Bardeen. Using the same model, Massar [45] recently showed that the emitted particles continue to have a thermal distribution with a time-dependent Hawking temperature $(8\pi M(t))^{-1}$. We refer readers to the latter work for details.

With this as a backdrop, the goal of our current program is to [46]
1) derive a fluctuation-dissipation relation embodying the backreaction for this quasistationary regime;
2) apply the stochastic field-theoretic formalism to the near-Planckian regime and derive an Einstein-Langevin equation for the dynamical metric including the effects of induced metric fluctuations.

4.2 Fluctuation-Dissipation Relation of Candelas and Sciama

On the first issue, historically Candelas and Sciama [3] first proposed a fluctuation-dissipation relation for the depiction of dynamic black hole evolutions. But as we will point out, their relation does not include backreaction in full and is not a FDR in the correct statistical mechanical sense.

As a starting point they considered the *classical* relation, due to Hartle and Hawking [47], between energy flux transmitted across the horizon of a perturbed black hole and the shear [3]:

$$\frac{d^2E}{dtd\Omega} = \frac{M^2}{\pi} \mid \sigma(2M) \mid^2, \tag{4.4}$$

where $\sigma(2M)$ is the perturbed shear of the null congruence which generates the future horizon.

In turn, the dissipation of horizon area with respect to the advanced null coordinate is related to the energy flux across the horizon, and the above equation becomes (see, for example [48])

$$\frac{dA}{dv} = 4M \int_H \mid \sigma \mid^2 dA, \tag{4.5}$$

the integral being performed over the horizon.

The classical formula above immediately suggests a fluctuation-dissipation description: the dissipation in area is related linearly to the squared absolute value of the shear amplitude. This description is even more relevant when the gravitational perturbations are quantized. Then the integrand of the right-hand-side of Eq.(4.5) is $\langle \sigma^* \sigma \rangle$, the expectation value being taken with respect to an appropriate quantum state. Candelas and Sciama choose this state to be the Unruh vacuum, arguing that it is the vacuum which approximates best a flux of radiation from the hole at large radii.

The details of this expectation value are given in [3]. Here we simply note that, with the substitution of this quantity in (4.5), the left-hand-side of that equation now represents the dissipation in area due to the Hawking flux of gravitational radiation, and the right-hand-side comes from pure quantum fluctuations of gravitons (as opposed to semiclassical fluctuations of gravitational perturbations, which are induced by the presence of quantum matter). It is tempting to regard this as a quantum FDR characteristic of the Hawking process, as do the authors of [3].

However, Eq.(4.5) is not a FDR in a truly statistical mechanical sense because it does not relate dissipation of a certain quantity (in this case, horizon area) to the fluctuations of *the same quantity*. To do so would require one to compute the two point function of the area, which, in turn, is a four-point function of the graviton field, and intimately related to a two-point function of the stress tensor. The stress tensor is the true “generalized force” acting on the spacetime via the equations of motion, and the dissipation in the metric must eventually be related to the fluctuations of this generalized force for the relation to qualify

[3]The analysis of Candelas and Sciama holds for a Kerr black hole. However, we simplify this to the Schwarzschild case in the interest of clarity; no qualitative features are lost in this simplification.

as an FDR. The calculation of 4-point functions of the metric perturbations h_{ab} and the correct FDR is currently under investigation [41, 46]

4.3 Bekenstein's theory of black hole fluctuations

The importance attributed to the correlation of mass function was a central point in Bekenstein's theory of black hole fluctuations [49]. Because it bears some similarity in conceptual emphasis to our approach we'd like to refresh the reader's memory of this work. We will also comment on the basic difference from our approach.

Bekenstein considered the mass fluctuations (and fluctuations of other parameters) of an *isolated* black hole due to the fluctuations in the radiation emitted by the hole, and considers the question of when such fluctuations can be large. For simplicity, only mass fluctuations are considered. The basic assumption is that the black hole mass $M(t)$ is a stochastic function with some probability distribution due to the stochastic emission of field quanta, and furthermore, that energy is conserved during the stochastic emission of quanta. As we shall see, this latter assumption leads to some startling predictions. With these assumptions, one may express the black hole mass at some time $t + dt$ in terms of the mass at an earlier time t by the equation

$$M(t + dt) = M(t) - m(dt), \tag{4.6}$$

where $m(dt)$ is the energy taken away by radiation in time dt. Averaging the above equation yields

$$\begin{aligned} \frac{d}{dt}\langle M\rangle &= -\frac{m(dt)}{dt} = -\langle L\rangle \\ &= -\alpha\hbar\langle M^{-2}\rangle, \end{aligned} \tag{4.7}$$

where Eq. (4.1) with the Hawking luminosity is used here. Furthermore, squaring Eq. (4.6) before taking the average yields,

$$\frac{d}{dt}\langle M^2\rangle = 2\alpha\hbar\langle M^{-1}\rangle + \beta\hbar^2\langle M^{-3}\rangle, \tag{4.8}$$

the second term in the above expression being obtained by Bekenstein from an approximate expression for the energy distribution of quanta emitted in time dt.

Defining $\sigma_M = \langle M^2\rangle - \langle M\rangle^2$, and using the moment expansions

$$\begin{aligned} \langle M^{-1}\rangle &= \frac{\langle M^2\rangle}{\langle M\rangle^3} + \mathcal{O}(\langle M^3\rangle) \\ \langle M^{-2}\rangle &= -\frac{2}{\langle M\rangle^2} + \frac{3\langle M^2\rangle}{\langle M\rangle^4} + \mathcal{O}(\langle M^3\rangle) \\ \langle M^{-3}\rangle &= -\frac{5}{\langle M\rangle^3} + \frac{6\langle M^2\rangle}{\langle M\rangle^5} + \mathcal{O}(\langle M^3\rangle), \end{aligned} \tag{4.9}$$

Eqs. (4.8) and (4.7) together imply

$$\frac{d}{dt}\sigma_M = \frac{\hbar}{\langle M\rangle^3}\left(-4\alpha\sigma_M + \beta\hbar(1+6\sigma_M)\right). \tag{4.10}$$

The above differential equation for σ_M possesses the approximate solution

$$\sigma_M \sim \chi\,\hbar\left(\frac{M_0^4}{\langle M\rangle^4} - 1\right), \tag{4.11}$$

χ being a constant related to α and β, and M_0^4 an integration constant, to be interpreted as the mass when it is sharply resolved (i.e. when $\sigma_M = 0$). As pointed out by Bekenstein, one of the many consequences of this solution is that the fluctuations σ_M can grow as large as $\langle M\rangle^2$ for $\langle M\rangle = M_c \sim \hbar^{1/6}M_0^{2/3}$. According to this picture, therefore, depending on the initial mass, mass fluctuations can grow large far before the Planck scale. Once the critical mass M_c is reached, the dynamics of the black hole differs drastically from the standard semiclassical picture, because the equation for $\langle M\rangle$, Eq. (4.7), is itself driven by the fluctuations in mass.

A crucial assumption for the validity of such a scenario for black hole evaporation is, of course, the stochastic energy balance equation (4.6) and the related Eq. (4.7). It is possible that different assumptions for the stochastic dynamics lead to drastically different conclusions about the late stages of the evaporation process. The prediction of Bekenstein's theory is at variance with Bardeen's in which the semiclassical picture of black hole evaporation remains valid until the hole reaches Planck size. Our stochastic semiclassical gravity theory would also support Bardeen's scenario as the fundamental stochastic (Einstein-Langevin) equation for the black hole mass is expected to take the form

$$\frac{dM}{dt} = -\frac{\alpha\hbar}{\langle M\rangle^2} + \xi(t), \tag{4.12}$$

where $\xi(t)$ is a stochastic term with vanishing average value. If such an equation were to hold, the average mass would be independent of the fluctuations, while the fluctuations would be slaved to the dynamics of the average mass, where backreaction will be incorporated in a self-consistent manner. As we showed in Sec. 2 this type of behavior of the mean field (semiclassical metric) and fluctuations is also found to occur in the treatment of backreaction in cosmological spacetimes.

We anticipate an Einstein-Langevin equation of the form (4.12) for the description of black hole evaporation in contrast to Bekenstein's equations of the form (4.7). The Einstein-Langevin equation when averaged over the probability distribution for the noise ξ should yield a semiclassical Einstein equation of the Bardeen type as a mean field theory.

5 Prospects

Here we have discussed some representative work related to ours on semiclassical black hole fluctuations and backreaction (other noteworthy proposals include that of 'tHooft [50], and

work on 2D dilaton gravity [51]). We have also sketched our approach, and marked out some important points of departure. This include 1) metric fluctuations and their role in backreaction – our definition of (induced) metric fluctuations is in terms of graviton 4 point functions; 2) the true statistical mechanical meaning of the fluctuation-dissipation relation and its embodiment of the backreaction effects. Our formulation testifies to the existence of a stochastic regime in semiclassical gravity where the dynamics of spacetime is governed by an Einstein-Langevin Equation which incoporates metric fluctuations induced by quantum field processes. We wish to explore the physics in this new stochastic regime, including possible phase transition characteristics near the Planck scale and its connection with low energy string theory predictions.

For the implementation of this program currently we are engaged in
a) setting up the CTP effective action for the quasistatic and dynamic cases [42, 46],
b) computing the fluctuations of the energy momentum tensor for fields both in the Hartle-Hawking state and the Unruh state [41], and
c) exploring the interior solution of a collapsing black hole [52] as this bears closer resemblance to a cosmological problem (Kantowski Sachs universe) [36].

This program will take a few years to fruition and we hope to report on some results in Vishu's 65th birthday celebration.

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